International Journal of Mathematical Education in Science and Technology, 2015
Vol. 46, No. 2, 187–204, http://dx.doi.org/10.1080/0020739X.2014.951007

A dynamic applet for the exploration of the concept of the limit of a sequence

Kell Cheng[a*] and Allen Leung[b]

[a]*Department of Mathematics and Information Technology, Hong Kong Institute of Education, Hong Kong, P.R. China;* [b]*Department of Education Studies, Hong Kong Baptist University, Department of Education Studies Hong Kong, P.R. China*

(*Received 13 December 2013*)

This paper reports findings of an explorative study that examine the effectiveness of a GeoGebra-based dynamic applet in supporting students' construction of the formal definition of the limit of a sequence or convergence. More specifically, it is about how the use of the applet enables students to make connections between the graphical representation and the formal definition. There are several core components that one can extract from the definition and the applet is specifically designed to help students with the components. Relevant student data from the study are presented with respect to these components. The significance of the applet and further research directions are discussed at the end with reference to existing research.

Keywords: dynamic applet; limit; sequence

1. Introduction

The limit concept has always been a difficult topic for upper secondary and university students. Studies indicated that the concept of the limit of a sequence has significant impact on other related concepts in advanced mathematics.[1–3] Due to the complex logical structure of the rigorous definition of limit of sequence,[3–5] it is not surprising that students often fail to understand the limit concept and they usually resort to rote memorization of the rigorous definition. Formally stated, L is the limit of a sequence $\{a_n\}$ if for any positive real number ε, there exists a natural number N such that for every natural number n greater than N, the absolute distance between a_n and L is less than ε. Symbolically, this is usually written as

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that for } n > N,\ |a_n - L| < \varepsilon.$$

In this paper, we will call the above statement the $\varepsilon - N$ definition. From this definition, one could imagine a dynamism that exists between the quantities involved; the variation of one quantity is dependent on or driven by the variation of the other. Research studies suggest that it is not straight forward to relate dynamic concept with formal definitions.[1,6] According to Tall and Vinner,[1] a *concept image* consists of all the cognitive structures in the individual's mind that are associated with a given concept which has evolved over time, whereas *formal concept definition* is based on a discourse commonly accepted by the mathematical community at large. Students' conception of a formal definition is usually

*Corresponding author. Email: khfcheng@ied.edu.hk

influenced or even dominated by their own previously built-up concept image.[1,3,7,8] Swinyard [8] further conjectured that students are more likely to use a concept image of limit associated with the previous learning experiences to elucidate the formal definition, rather than to develop a concept image of limit based on the formal definition. Hence, it might be more pedagogically conducive to help students to develop a comprehensive and structured concept image of limit before the introduction of the formal $\varepsilon - N$ definition.

In a traditional classroom setting, teachers often introduce the concept of the limit of a sequence by expressing it in a dynamical way as where the sequence 'is tending to' or 'getting closer to'; however, this only leads to a partial conception of the concept image formed which may hinder students' understanding towards the $\varepsilon - N$ definition when they come across it formally.[1] In particular, these expressions may easily lead to students' misconception towards the concept of the limit of a sequence as motion-oriented, unreachable [9] and even mixed up with other concepts like asymptotes or cluster points.[10,11] Cottrill et al. [6] believed that students' failure to move from informal to formal conception of limit is practically a result of insufficient development of a strong dynamic image. Tall and Vinner [1] and Roh [10] suggested that there are cognitive dissonance to students' concept image and concept definition. Students were confused when introduced with the formal definition as it might not be coherent with the images they already constructed in their mind. Students usually start learning the limit concept with monotone sequences, when they are asked to handle constant sequences or oscillating divergent sequence, they are confused due to the discrepancy.

To effectively teach the $\varepsilon - N$ definition, one needs to realize the various aspects and difficulties that students encounter in the learning process. This definition has a complex structure and there are discussions on this particular topic in the literature.[3,4,5,12] Cory and Garofalo [5] discussed the complexity involved in the $\varepsilon - N$ definition and listed out aspects that students must conceptualize in order to grasp the idea. These aspects include

(1) the absolute inequality,
(2) the purpose of N,
(3) the use of the quantifiers ($\forall$, $\exists$) and
(4) the relationship between ε and N.

In this study, we take an exploratory approach in which students are to learn the rigours concept of the limit of a sequence by constructing an equivalent form of the definition themselves. The key to this approach is that students are not told of the definition or its key attributes or core components in advance; instead, they are to go through a series of exploratory tasks on the core components:

(1) the absolute inequality,
(2) the purpose of N and
(3) the dependency of ε on N.

Note that the component on quantifiers in Cory and Garofalo's work is not part of our exploratory tasks as it is implicitly embedded in the relationship between ε and N and as students construct their own versions of the definition, they will use the ideas of the quantifiers, albeit not writing them out explicitly.

There are two key ingredients here. Students are provided with a virtual dynamic GeoGebra-based applet and a set of guidance worksheets (see samples in Appendix 1) for the exploratory tasks. The underlying design of the applet is that it is a tool that students can use to visualize and manipulate the various core components from the $\varepsilon - N$ definition. The applet (tool) by itself does not provide guidance on how to bring about the $\varepsilon - N$ definition; rather it empowers students to discern critical ideas to make sense (the concept image) of the convergence of a sequences and thus construct a version of the $\varepsilon - N$ definition (formal concept definition).

2. Technology-enhanced dynamic visualization

Plass, Homer and Hayward [13] reviewed research on learning from dynamic visual representations and confirmed that effective visualization has a strong impact on learning. They presented a design principle on dynamic visualization, namely, *principle of manipulation of content* which suggests that learning from visualization is improved when learners are able to manipulate the content of a dynamic visualization compared to when they are not able to do so. Kidron and Zehavi [14] incorporated dynamic Mathematica© graphs to visualize the process of convergence. Evidences showed that dynamic software provides significant assistance in helping students to connect visual and algebraic expressions to the formal definition of limit.

Cory and Garofalo [5] designed dynamic sketches to study the limit of a sequence using Geometer's Sketchpad©. Unlike the dynamic graphics designed by Kidron and Zehavi [14] that required entering commands to see the changes in the graphic, students could drag the graphical elements in these dynamic sketches to desired positions. These dynamic sketches not only visualize the images of the sequences, but they also allow students to enter values, zoom-in and zoom-out as well as display the limit's numerical value. Cory and Garofalo showed results on three pre-service teachers' perception of the limit of a sequence by asking them to explain the $\varepsilon - N$ definition with reference to its visual representation on the dynamic sketches.

3. The design of a dynamic applet

With reference to the research studies discussed in the previous sections, we designed a GeoGebra-based dynamic applet that simulates a virtual version of the ε-strip by Roh [3,10,11] and Roh and Lee,[15] which is equipped with more versatility. It is intended to be a dynamic visualization platform for students to explore the meaning and possible definitions of the limit of a sequence.

Leung [16] proposed a techno-pedagogic task design principle that focuses on a pedagogical process 'in which learners are empowered with amplified abilities to explore, re-construct (or re-invent) and explain mathematical concepts using tools embedded in a technology-rich environment.' [16, p.327] Such pedagogical process consists of three nested epistemic modes: establishing practices mode, critical discernment mode and situated discourse mode. These modes concatenate a knowledge acquisition process starting from the use of a technological tool to the development of a mathematical discourse with respect to the tool. In particular, when designing a technology-enhanced pedagogical environment for mathematical content, critical aspects should be embedded into the 'how' of using the tools involved. Take the present case of designing a dynamic applet to explore the rigorous definition of the limit of a sequence. There are three critical aspects in the $\varepsilon - N$ definition:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that for } n > N,\ |a_n - L| < \varepsilon.$$

C1. The geometric meaning of the inequality $|a_n - L| < \varepsilon$
This inequality is an essential part of the definition that captures the concept of convergence. The other components in the definition are conditions under which this inequality holds. A visualization tool that can lead to the intended definition should graphically represent this inequality in a way such that students can manipulate it as a dynamic measuring tool which can be used to discern the idea of convergence of a sequence.

C2. The condition for n to be larger than N ($n > N$)
This is a condition that should be embedded in the design of the dynamic measuring tool mentioned in C1. Students are often confused about what n and N represent since N does not explicitly enter the inequality and n relates to the part of the sequence that comes after N, where N is implicit in the inequality $>$ once an ε is decided. Students should be able to freely manipulate the dynamic measuring tool to find N so that ascertaining the existence condition $\exists N$ becomes a critical visual cognitive activity for students during the exploration process.

C3. The dependency between ε and N ($\forall \varepsilon > 0, \exists N$)
The dependency of N on the choice of is the most difficult, yet critical, aspect for students to grasp. The dynamic measuring tool should allow students the flexibility to explore and explain possible dependency relationship between ε and N in the process of reconstructing (or re-inventing) the limit definition. It may be pedagogically more challenging not to embed rigidly the intended dependency relationship into the dynamic measuring tool. This will make the exploration more open for critical discernment.

Figure 1 depicts a snapshot of the GeoGebra dynamic applet (henceforth it will be called the applet) for the sequence $a_n = 10\,(\sin n/n) + 4$. In the applet, there is a dynamic

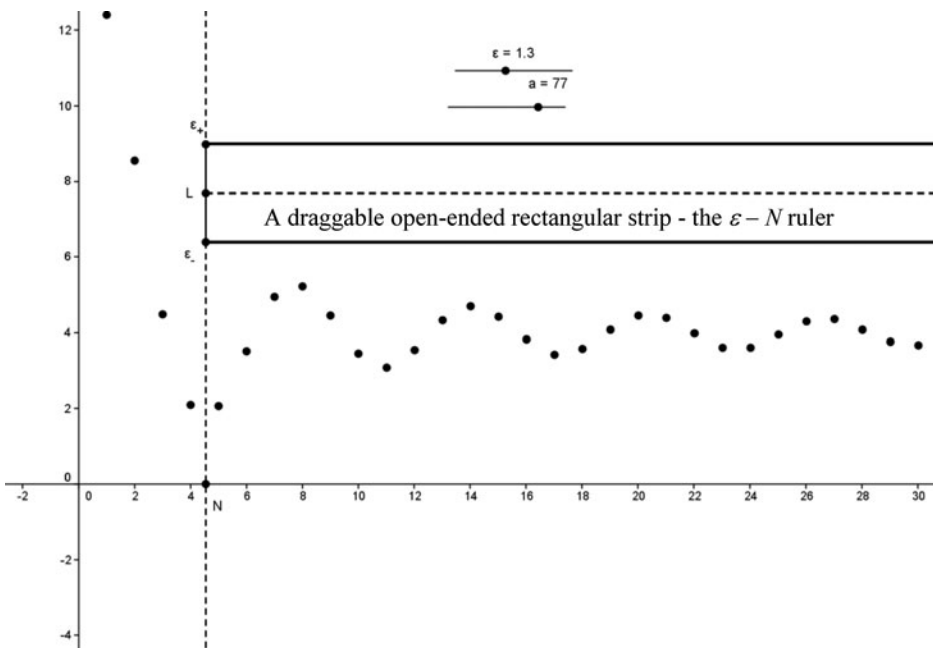


Figure 1. Snapshot of the GeoGebra dynamic applet (henceforth it will be called the applet) for the sequence $a_n = 10\,(\sin n/n) + 4$.

rectangular strip that opens 'ad infinitum' to the right. The position of this rectangular strip is controlled by a draggable point L. That is, this open-ended strip can be moved to anywhere on the sketch as one likes by dragging L to different positions. The width of the strip is adjustable by an ε-slider with the ε value ranging from 0 to 2 incremented by 0.1 (these values can be changed easily). A horizontal dotted ray radiates from L to the right that splits the strip into two equal upper and lower halves with each half having length ε (as indicated by the marks ε_+ and ε_-). A dotted line perpendicular to the x-axis through L is attached to strip and its intersection with the x-axis is denoted by N. We will call this open-ended rectangular strip the $\varepsilon - N$ ruler. An a-slider with integer value ranging from 0 to 1000 (this range can be changed easily) controls the number of sequence terms to appear in the applet. Press and hold the Shift button which activates the zooming mode for the whole applet via scrolling, the moving of the whole applet by dragging, or the scaling for the x- and y-axes. The applet is converted and modified to a web-based applet[1] where student can type in a desired sequence (see Figure 2).

The $\varepsilon - N$ ruler represents the inequality. The dynamic width of the ruler is determined by choices of ε, the left vertical end of the ruler marks the location (existence) of possible N's, and the right open end of the ruler captures the meaning of for all $n > N$. The dynamism of the $\varepsilon - N$ ruler is designed by taking into consideration the three critical aspects C1 to C3 discussed above. Students are free to choose the ε value and to locate N by moving the ruler. Thus, the dependency between ε and N is not embedded in the dynamism, but it should be a critical discernment that students need to be aware of under properly guided

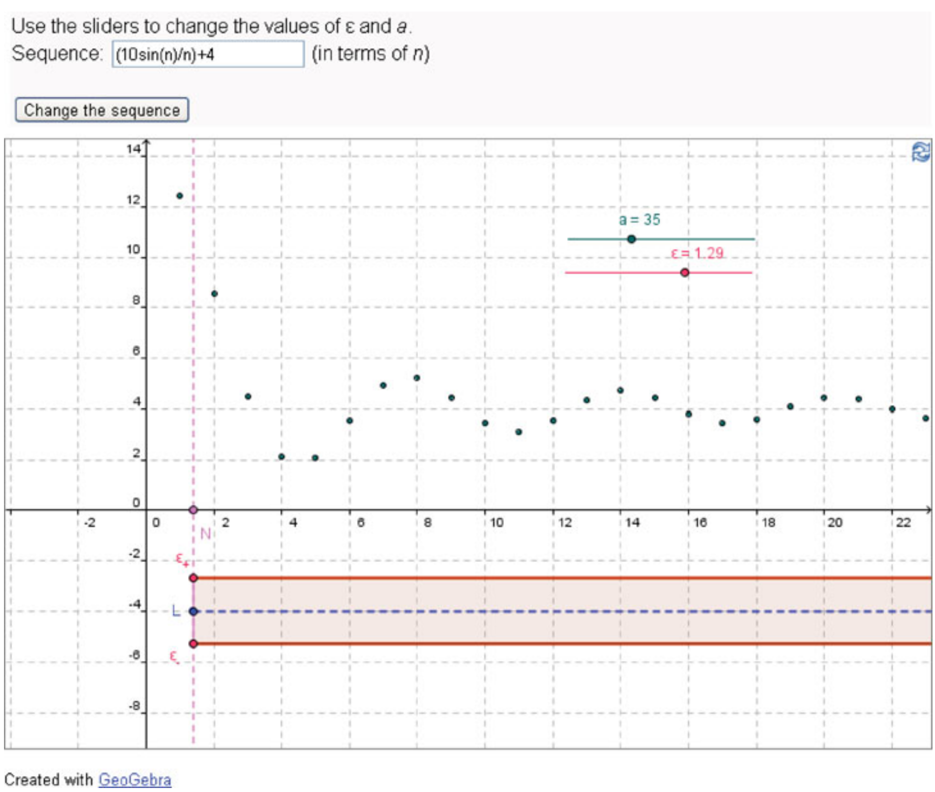


Figure 2. Web-based applet used by the students in which they can type in a desired sequence.

discovery activities. There are at least two possible operational procedures to guide the use of the $\varepsilon - N$ ruler:

P1. Choose an ε, then move the strip (hence the N value) so that all the sequence points to the right of N fall inside the strip.
P2. Move the strip (hence the N value) to a desired position, then choose an ε so that all the sequence points to the right of N all fall inside the strip.

A critical discernment is that when the sequence under studied converges, both P1 and P2 work for any ε and N always; but when the sequence diverges, P1 and P2 might or might not work for any ε and N. The pedagogical content of the applet thus lies on building up enough epistemic experience for students to contrast the behaviours for different types of sequence using the $\varepsilon - N$ ruler, hence to decide whether P1 or P2 would ascertain convergence, which in turn gives hint to the ε and N dependency.

4. An explorative study

An explorative study was carried out to examine the effectiveness of the applet in helping students to construct the formal definition of the limit of a sequence or convergence. The applet and a sequence of guidance worksheets (Appendix 1) were implemented in an introductory analysis course at a teacher's training institute in Hong Kong. Students in this course had prior exposure of calculus and limits but not the $\varepsilon - N$ definition. The worksheets were designed to lead the exploration of the three components C1, C2 and C3.

Individual task interviews were conducted to gather feedback and data for analysis. Invitations to the interviews were extended to all 15 students in the class and 7 of them agreed to participate. Each interview ran approximately 30 minutes in which a student was to first independently work through a set of worksheets (Appendix 1) with the applet. The questions were

(1) to explore the existence and the value of $\lim_{n\to\infty} \frac{10\cos(n)}{n}$,
(2) to discuss the idea of 'tending to infinity' and
(3) to examine the dependency relationship between ε and N.

And then the student was asked by an interviewer to explain the reasoning behind his/her work. The interviewer was the research assistant of this study and was not known to the students. Each interview was videotaped and transcribed afterwards.

In what follows, we present the findings in these interviews with respect to these three aspects.

4.1. *Existence of limit*

From the interviews, it was observed that students took the following general procedure to find the limit using the applet.

(1) Adjust the a-slider to its maximum, in this case, 1000.

(2) With or without adjusting the ε value, place the 'shaded region/box' (using students' choice of words, they mean the $\varepsilon - N$ ruler) to a position where the tail-end (the part of the sequence that comes after the dotted vertical line attached to the $\varepsilon - N$ ruler) of the sequence falls inside the $\varepsilon - N$ ruler and take note of the N value.

(3) Diminish the ε value and push forward the $\varepsilon - N$ ruler to the right (not necessarily in this order) to get a narrower $\varepsilon - N$ ruler that covers the tail-end.

All students guessed (or realized) that the limit of $\lim_{n\to\infty} \frac{10\cos(n)}{n}$ was zero, and followed the general procedure above to find (or to confirm) the limit. Most students began by placing the limit, represented by the line L, on the x-axis (since the guess was 0) at an arbitrary N position and dragged to the end of the sequence to observe the trend, then they reduced the width of the $\varepsilon - N$ ruler by reducing the value of the ε. Next, they adjusted the position of the $\varepsilon - N$ ruler to a suitable starting point N so that it could cover all the points of the sequence after the Nth term. Figure 3 depicts an example of a student's work on the applet.

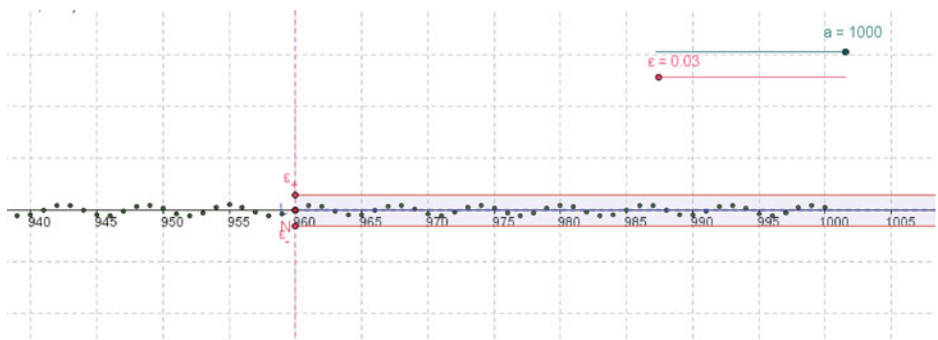


Figure 3. An example of one of the students' work on the applet.

The seven students had three types of applet-based explanation to discuss what they have discovered.

Explanation 1: Manipulation for visual verification

Students used the dynamic features of the applet to visualize and verify the existence of limit. Students 1, 3, 5 and 6 described the trend of the sequence to convince the interviewer that the sequence was moving towards zero. Some mentioned that the sequence oscillated vigorously for the first few terms and the oscillation became smaller as n got larger. Two of them described the nature of the end part of the sequence by 'the sequence will finally become a line' (Student 1) and 'the sequence will become more stable' (Student 6). Nevertheless, they all confirmed that the limit for the sequence was zero. Essentially, they found an N for a small ε where the 'tail' could be covered by the shaded region (the $\varepsilon - N$ ruler) with the left edge anchored at N when they put the line L on the x-axis. As the seven reasoning are fairly similar, we present two of them in the following (Figures 4 and 5):

Work by Student 1:

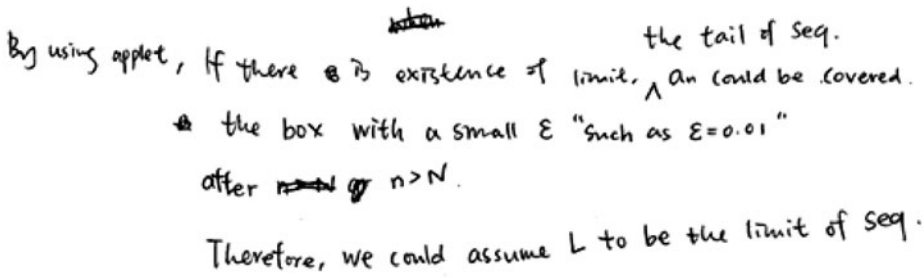


Figure 4. Student 1's response to Question 1.

Work by Student 6:

> 'Should be . . . this distance that is surrounded by ε . . . Between the two red lines represents ε positive and negative, also this two red lines contains the tail of the sequence, $a_n - L$ should be the distance by subtracting this highest point (pointing y to a local maximum point in the sequence) with a lowest, ε is larger than this point subtracts this point.'

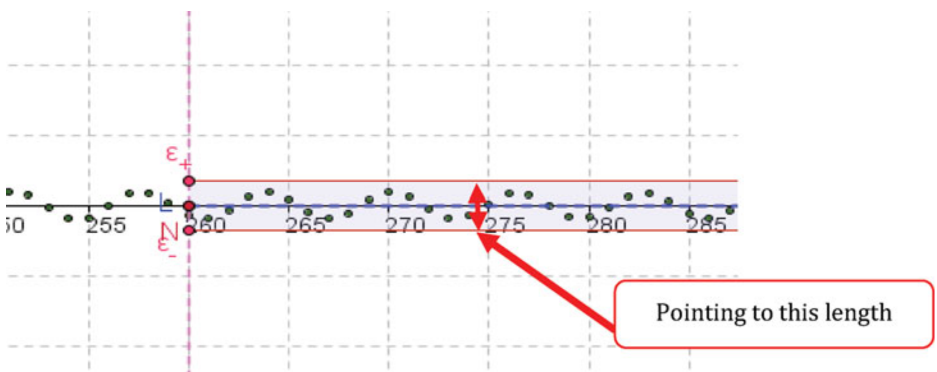


Figure 5. Student 6's response to Question 1.

The ε -slider and a-slider were instrumental in students' exploration process. Student 5 explained that

> 'This frame (shaded region) can surround it ($a_n - L$), from this point L. . . (shaded region) can contain all points that come after, and then by keep on decreasing (ε), limit can be found . . . the further away (ε) becomes smaller and smaller.'

However, he was uncertain about the limit during the interview when the number of terms was set to 120, 315 and 500 with ε varied from 0.2 to 0.08. He was not able to place the shaded region to a suitable position such that all points at the 'tail' can be covered. The student explained that the trend of the sequence was not obvious by considering the first 500 terms only. Once he set the number of terms shown to 1000, he was able to quickly figure out the limit by applying the general procedure.

Explanation 2: Uncertainty and doubt

Instead of merely a graphic computational tool, the applet was designed to be a cognitive thinking tool for students to develop mathematical reasoning. In particular, rather than giving students 'correct' and 'exact' answer to the value of the limit, students have to makes guesses. This allowance of uncertainty and doubt provide students epistemic situations where routine procedure can be transformed into conceptual understanding. The following episode illustrates a taste of this kind of cognitive applet usage. For students 2 and 7, rather than attending to the trend of the sequence, they picked different values for ε and dragged the $\varepsilon - N$ ruler to suitable positions. Both students repeated this step a few times and found several pairs of ε and N when putting the line L on the x-axis (Figure 6).

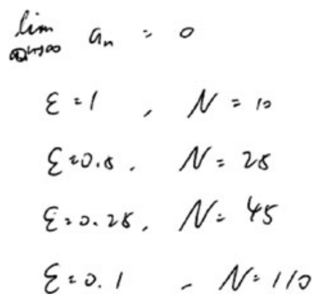
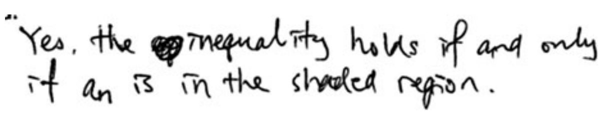


Figure 6. Student 7's choices of ε, N pairs.

Since they could find many pairs of ε and N satisfying the condition, they concluded that the limit of the sequence is zero. However, student 7 questioned the accuracy of the limit found by using the applet. He demonstrated that if he moved the line L in the applet slightly below zero, the 'tail' could still be covered by the shaded region with the same pair of ε and N; hence he had reservation on whether the limit he found was the correct one.

Explanation 3: Re-scaling

Another kind of epistemic situation that the applet can provide is where students have freedom to choose how to represent the sequence by re-scaling the x- and y-axes. Student 4 related the formal $\varepsilon - N$ definition to the applet to explain the existence of limit. He mentioned that as he set ε to a positive value, by re-scaling the axes, he saw in the applet that all points of the sequence after the 500th terms were covered by the $\varepsilon - N$ ruler. To him this meant that the distance from the a_n's to the limit were smaller than ε, that is, $|a_n - L| < \varepsilon$. Hence by definition, what he saw confirmed that the limit was the L in the applet:



'When I take L as 0 with N bigger than 500, all the a_n afterward subtracting the limit value will be smaller than ε, therefore I take 0 as the limit.' (student 4)

Figure 7 is a screen shot of student 4's sequence with 0.01 as one unit for the y scale and 50 as one unit for the x scale. This choice of scales gave the sequence a convincing look of convergence to the x-axis. Notice that N is at a relatively small value close to 0. In particular, it gives a strong graphical sense for Roh's ε-strip definition B which emphasizes that there are only finitely many sequence points that are outside the ε-strips.

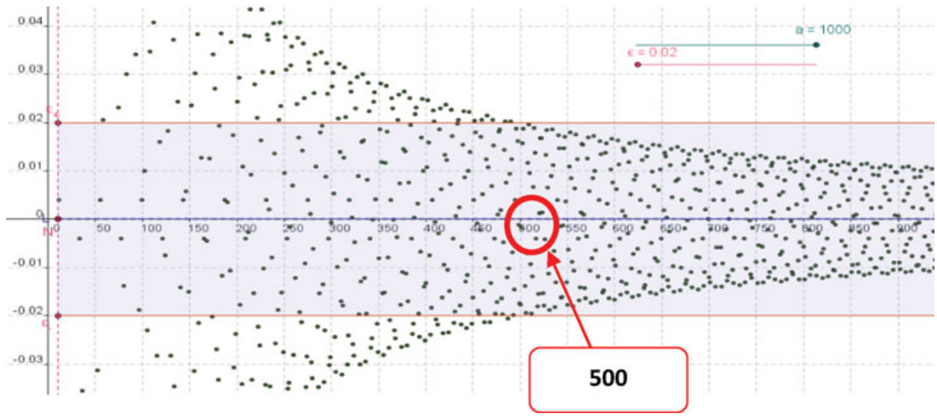


Figure 7. Screen shot of student 4's sequence with 0.01 as one unit for the y scale and 50 as one unit for the x scale.

These student responses show that the $\varepsilon - N$ ruler was instrumental in helping students to visualize and interpret critical aspect C1 (the geometric meaning of the inequality). The $\varepsilon - N$ ruler acted as a measuring tool but did not provide any hint on what the limit should be. Student 7's uncertainty about the real value of the limit revealed the design intention of the $\varepsilon - N$ ruler, that is, it should only become a student's cognitive instrument to think about the existence and value of the limit rather than be a tool to pin point an answer for the limit. Moreover, the $\varepsilon - N$ ruler does not impose dependency relationship between ε and N. It is entirely up to the students to decide on which one to vary first. For the sequence that students explored during the interview, in general the focus was not on the order of variation for ε and N, but on whether the $\varepsilon - N$ ruler would cover the tail end (critical aspect C1).

4.2. *Concept of tending to infinity*

During the interviews, students were asked to explain how the concept of infinity got into the $\varepsilon - N$ definition. All students recognized that the key components relating infinity to the definition were the variables n and N. Students realized the importance of N as a cut-off point such that all terms after N should satisfy the inequality. This was evident during the interviews as students kept adjusting the edge of the $\varepsilon - N$ ruler to make sure the tail end of the sequence is covered inside.

Students 1, 2, 3, 4 and 7 explained that the condition $n > N$ in the $\varepsilon - N$ definition expressed the idea of infinity. They indicated that the N could be a certain number represented by the edge of the $\varepsilon - N$ ruler, and whenever there was an n greater than N and all points left behind could be covered by the $\varepsilon - N$ ruler, the sequence should converge. They indicated that the n in the definition was not specified and could be any number greater than N, thus n can be 'infinitely large':

> 'I want to focus on seeing which position, from that position it begins to cover . . . when I move to this position, I find that the part after this position is . . . is bounded.'(student 2)

> 'Actually (putting it in other position) is OK, I want to try to put it to a point that is a bit further down, because ... I just tried with (N equal) eight hundred something, when I put ε to very small also can cover those points (a_n), but, I want to see what would happen further ahead.' (student 3)

Student 5 used the applet to help him explain the concept of infinity during the interview. He said the n can be any point on the x-axis, he pointed to two positive values on the x-axis explaining that he could either choose these two points or any other point on the x-axis, so it was 'reasonable' to choose an n which was infinitely large. Student 6 just considered the condition of N to be a natural number and said that we could take any natural number as large as we wanted, which contained the case of infinity. When using the applet in the interview, these two students realized the importance of finding a starting point N such that the 'tail' could be covered by the $\varepsilon - N$ ruler, nevertheless, both of them did not consider the condition $n > N$ as a whole and failed to connect the general procedure to use the applet with the concept of infinity.

Critical aspect C2 (the condition $n > N$) is a primitive for the concept of tending to infinity and the open-endedness of the $\varepsilon - N$ ruler was designed to capture this idea. N is explicitly identified with the edge of the $\varepsilon - N$ ruler and n is implicitly hidden in a-slider. Visual expressions like 'the (open) tail-end of the sequence', 'push forward N a little' that students often used reflected a perception of 'as large as one likes' and students were aware that if the a-slider was not set to a large number, for example at least 1000, they might not be able to determine the limit.

4.3. *The ε and N dependency*

Most students emphasized the importance of ε in the applet. They realized that the ε value was essential in verifying the limit of the sequence. Student 3 suggested that by decreasing the ε value, it would indicate whether the line L in the applet needed to be adjusted. Students 4 and 7 further explained that if an N was chosen first, points of the sequence beyond the Nth term were still distracted from each other, so it was more reasonable to fix a range for the shaded region that was considerably small first, then the limit found would be more reliable. For the dependency between ε and N, student 1 thought that ε depended on N:

> I think ε is depends on N. We may not find an ε for some N ... when N is getting larger, ε can be getting small. (student 1)

Student 2 was convinced that ε needed to be determined first and expressed a causal relationship between ε and N for convergent sequences (Figure 8).

However, the explanation for this dependency did not reflect an understanding of convergence:

> I feel I would decide on ε first, because the value of ε is conditional, it must be bigger than 0, and cannot be too large, that is, it needs to be close to 0, therefore decide on ε first, then find a corresponding N. (student 2)

Student 3 had a dilemma after he used the applet. Originally, he thought that he should choose an N before finding a corresponding ε. However, when using the applet, he was more 'comfortable' not to be preferential in the choice of N or ε in determining the limit:

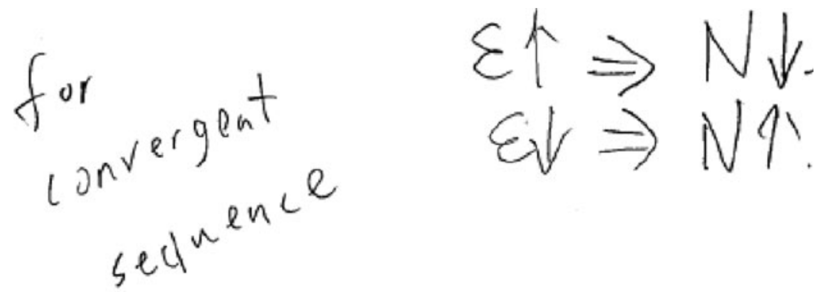


Figure 8. Student 2's understanding on the ε and N dependency.

> When N becomes larger, ε can become small, and vice versa ... for the sake of reasoning, I feel that Nshould come first, but ... when using this applet, I feel both ways are OK .., .., but on a second thought, it should be N first and then ε. (student 3)

Students 4, 5 and 6's perceptions were ambiguous. Even all of them finally indicated that ε should be determined first, their explanations showed a wavering between the two possible dependencies.

> I think N depends on ε ... we saw that as ε decreases, N increases, however, no matter how N changes ... there is always a value of ε such that all the point with n greater than N are contained in the shaded region ... When ε gets bigger, N gets small ... I feel ε should be chosen first. (student 4)

> When N gets bigger and bigger, ε gets smaller and smaller, the shaded region can then surround all the points ... Mm ... should find ε first, because however small ε is, I can find a point N to make a_n on the box. (student 5)

> If when N is small, ε needs to increase ... if N is to cover the tail early, then ε needs to be big ... I feel that I can find ε first, because the sequence extends indefinitely; choosing an ε first will fix a range, and then we can look for a suitable N. (student 6)

However, student 4 did claim that if we fixed the N and then found a corresponding ε, we would fail to find the ε in some cases. Student 4 tried but failed to provide a counter example to support the argument and indicated that the sequence used during the interview was not suitable to illustrate this claim.

Only student 7 could give a more consisent explanation as to why N should depend on ε:

> When we try to adjust the ε to smaller values, we find that N will become larger so that $\{a_n, a_{n+1}, a_{n+2}, \ldots\}$ are in the region, this shows that the limit exists ... It is more fitting to fix ε first, then to find out N. If we set N first then to find ε ... say, we may set N to be 2, and then look for ε, in this case ε needs to expand a lot in order to cover the tail part, but look at this picture (in the Applet Figure 9) ... you can see that these points (a_n) are not centralized, therefore it would be difficult to find the limit. If we set ε first, the frame (shaded region) is adjusted to become small, you can still find the tail part ... the tail part will be comparatively more centralized around the limit, the centralized position is what we want to find ... (student 7)

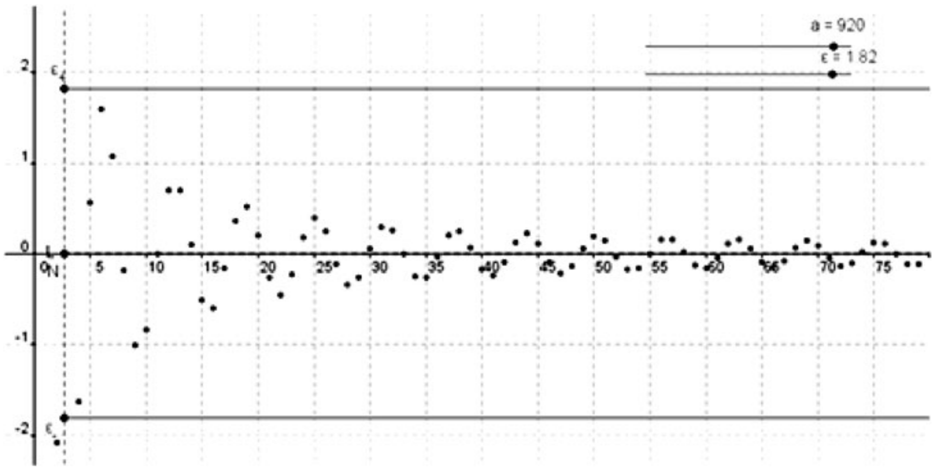


Figure 9. Snap shot of student 7's explanation on the dependency between ε and N using the applet.

It is worth noting that students 2 and 4 were able to use the applet consistently to demonstrate the ε and N dependency that is consistent with the $\varepsilon - N$ definition, even though their verbal explanation did not reflect a thorough understanding of their demonstration. They followed the procedure:

> Change the ε value (for all ε greater than zero) and adjust the starting point N to a suitable position (there exists a natural number N), so that all points afterward are covered by the shaded region (such that for all n greater than N, in inequality is satisfied).

The ambiguous explanations given by these students indicate that there is a gap between procedural skill and conceptual understanding when using the applet if students were not able to discern the necessary critical features.

These student responses confirm that the $\varepsilon - N$ ruler was indeed a measuring tool which students used to think and reason rather than used to get an answer (see students'

Table 1. Student perception patterns on the three components.

	Majority pattern	Minority pattern
Existence of a limit	Use dynamic features of the applet to verify the existence of limit. (students 1, 3, 5, 6)	For values of ε, choose suitable values of N so that for $n > N$. With enough pairs of ε and N found, students are convinced that the limit exists.(students 2, 4, 7)
Concept of tending to infinity	The idea of infinity is embedded in the n of $n > N$. As n has no specific values and it is positive, n can be infinitely large.(students 1, 2, 3, 4, 7)	The idea of infinity is embedded in the N of $n > N$. As N has no specified values and it is positive, N can be infinitely large.(students 5, 6)
Dependency of N on ε	Inconsistent explanation of the dependence of N on ε.(students 1, 2, 3, 4, 5, 6)	Consistent explanation of the dependence of N on ε.(student 7)

perception pattern in Table 1). This matches the rationale behind the design of the applet which allows two possible procedures (P1, P2) to measure the inequality. For a convergent sequence like the one student explored in the interviews, both P1 (N seems to depend on ε) and P2 (εseems to depend on N) are possible procedures to discover the limit. This was ascertained by students' ambiguous discourses when discussing the dependency between N and ε on a convergent sequence. Hence by just using the $\varepsilon - N$ ruler on a convergent sequence, one might not uncover the N and ε dependency in the $\varepsilon - N$ definition. In order to bring about the expected $N(\varepsilon)$ dependency, one may need to consider counter-examples like $a_n = \sin(n)$. In such a case, for any chosen N, one can always find an ε for $\varepsilon - N$ ruler to cover the tail, but the sequence does not converge. This implies P2 does not guarantee convergence. This discernment is critical to develop the $N(\varepsilon)$ dependency in the $\varepsilon - N$ definition. In what follows, we present a summarizing table for comparing majority and minority patterns of students' perception of the three components.

5. Discussion

The results of this study support the findings of Roh,[11] Cory and Garofalo [5] that students were able to articulate relationships between symbolic expression and (technology enhanced) graphical representation. In particular, the $\varepsilon - N$ ruler in the applet empowered students to manipulate the condition by concretizing the existence of N as placing (via dragging) the edge of the $\varepsilon - N$ ruler on an appropriate spot such that the $\varepsilon - N$ ruler becomes the visual representation of the absolute inequality in the $\varepsilon - N$ definition. All the seven students realized that this condition was essential to a definition for convergence by dragging the $\varepsilon - N$ ruler to cover the tail part of the sequence. This enactment shaped students' behaviour and interpretation during the interviews. It is consistent with Cory and Garofalo's [5] result that the implementation of dynamic software displays an explicit graphical representation of the phrase '$n > N$, $|a_n - L| < \varepsilon$'.

The applet was designed as a cognitive tool for students to develop mathematical discourses on limit of sequence like those presented in the previous sections. The feedback that the applet generated from students' interactions creates epistemic situations where discernment of critical mathematical ideas may occur. Different types of feedback produced by the dynamic elements in the applet, in particular the $\varepsilon - N$ ruler, support different ways to utilize the applet to explain the mathematical phenomena seen on the computer screen (for example, the three explanation types presented in Section 4.1). Under the framework of techno-pedagogic task design [16] mentioned in Section 4, the $\varepsilon - N$ ruler is a tool that facilitates the progression of the three nested epistemic modes: establishing practices, critical discernment and situated discourse.

In the establishing practices mode, students developed a general procedure to find the limit using the applet (like the one observed in the study) by refining their skills in scaling the axes to visualize the different 'faces' of the sequence, to place the L line to guess the limit, and other skills to use the $\varepsilon - N$ ruler to perceive the critical aspects C1, C2 and C3. These practices enabled students to shape their 'behaviours' and 'verbal expressions' (like 'shaded region', 'tail-end', 'cover', etc.) in using the applet.

When students were at ease with the above skills, they entered into the critical discernment mode by turning their focus of attention to a meta-level rather from a phenomenological level. In this mode, the $\varepsilon - N$ ruler became a meta-tool that students could use to ask questions like: What's the difference between the two procedures P1 and P2? What is a proper scaling to 'see' the convergence of the sequence? The guess for the limit L will never be exact using the $\varepsilon - N$ ruler, so what is the correct answer? What examples and counterexamples can be used to illustrate the dependency between ε and N? How does tending to infinity relate to covering the tail-end of the sequence using the $\varepsilon - N$ ruler? These were some critical questions considered by the students during the interviews and they prompted students to discern the logical structure of the $\varepsilon - N$ definition using the $\varepsilon - N$ ruler.

The answers to the above questions would initiate students into the situated discourse mode where they could 'talk about' the meaning of convergence using the '$\varepsilon - N$ ruler language' that they developed in the previous mode. In the explorative study even though students began to talk about convergence using the applet, this mode was not fully explored. We could have probed students deeper and asked them to formulate an $\varepsilon - N$ ruler definition for the convergence of sequence and compared it with the $\varepsilon - N$ definition.

Summarizing the findings in the explorative study supports that the $\varepsilon - N$ ruler has the potential to be transformed into a cognitive epistemic tool from a virtual manipulative tool under Leung's techno-pedagogic task design framework provided that students are given suitable tasks to guide them through the three epistemic modes. Thus, we could further study the types of pedagogy that are appropriate to achieve such tool transformation.

Under the theory of semiotic mediation framework,[17] the applet can be regarded as a digital artefact with a semiotic potential to facilitate a mediation process between the 'personal meanings emerging from its use to accomplish a task, and at the same time the mathematical meanings evoked by its use and recognizable as mathematics by an expert'.[18] The mathematics evoked by the applet that is recognized by an expert is the $\varepsilon - N$ definition, while for students, the ways to utilize the applet evolve into personal discourses like those described in Section 4. The applet is a cognitive instrument in the sense that it evokes known mathematical concepts, while at the same time supports development of mathematical concepts via student–applet interactive utilization scheme. The general procedure to find the limit using the applet outlined at the beginning of Section 4.1 is such a scheme. This scheme is not the intended learning outcome; it serves as an applet-based discourse to develop the limit concept. The evolution of utilization scheme and student discourse is not an impulsive process and the teacher needs to design appropriate tasks to scaffold its development. As suggested by Leung's pedagogic task design principle discussed in Section 5, sequence of classroom tasks using the applet can be designed to chart a course of teaching and learning starting with skill to use the applet, then to critical discernment, and finally to situated discourse. The applet was designed in such a way that students need to practice and develop skill to use it. In this process, decisions have to be made, and uncertainty and doubt will arise. This is when students ask 'What if . . .' questions and critical discernment comes about leading to applet-based discourse.

The data collected in the study shows that the applet is conducive to the perception of the formal $\varepsilon - N$ definition and that it has shown good epistemic and pedagogical potential. In retrospect, the semiotic potential of the $\varepsilon - N$ ruler should be more fully explored in the tasks using different types of convergent and divergent sequences in order to give students a critical contrasting experience to come up with 'personal definitions' of limit and convergence. The teacher could then orchestrate a collective mathematical discussion [19] using the $\varepsilon - N$ ruler to mediate between the personal definitions and the expert definition. This is consistent with the concept of the *didactic cycle* in classroom teaching and learning where activities with artefacts, individual production of signs and collective production of signs all 'contributing differently but complementarily to develop the complex process of semiotic mediation.' [18]

The next stage of research is to fine-tune the implementation of the applet along with guidance tasks in the classroom using suitable pedagogical framework and examine its effectiveness to bring about students' understanding of limit and convergence. The guidance tasks can be designed using the guided reinvention approach in which students are 'given the opportunity to experience a process similar to that by which a given mathematical topic was invented . . . tasks or sets of tasks should invite students to develop "their own" mathematics.' [20] In such a pedagogical move, task design [21] will play a crucial role to ensure the applet is a cognitive instrument to support students' propensity to ask 'What if. . .' questions. We anticipate that this direction of research will enrich the discussion on the use of virtual artefacts in the teaching and learning of analysis.[table-wrap]

Acknowledgements

The authors thank the referees for their insightful comments. Also, the authors would like to thank Alice Ho for her great assistance in the study.

Funding

This research was partially supported by the Teaching and Development Grant of the HKIEd [grant number FAS/TDG001/MIT/10-11 (T0089)].

Note

1. Web link to the Applet is http://www.math.ied.edu.hk/tdg2010_11/geogebra/2011/limit.html.

References

[1] Tall D, Vinner S. Concept image and concept definition in mathematics with particular reference to limits and continuity. Educ Stud Math. 1981;12(2):151–169.
[2] Tall D. The transition to advanced mathematical thinking: functions, limits, infinity and proof. In: Grouws DA, editor. Handbook of research on mathematics teaching and learning. New York, NY: Macmillan; 1992. p. 495–511.
[3] Roh KH. An empirical study of students understanding of a logical structure in the definition of limit via the ε-strip activity. Educ Stud Math. 2010;73:263–279.
[4] Mamona-Downs J. Letting the intuitive bear on the formal; a didactical approach for the understanding of the limit of a sequence. Educ Stud Math. 2001;48:259–288.
[5] Cory B, Garofalo J. Using dynamic sketches to enhance preservice secondary mathematics teachers' understanding of limits of sequence. J Res Math Educ. 2011;42(1):65–96.
[6] Cottrill J, Dubinsky E, Nichols D, Schwingendorf K, Thomas K, Vidakovic D. Understanding the limit concept: beginning with a coordinated process schema. J Math Behav. 1996;15(2):167–192.
[7] Pinto M, Tall D. Student constructions of formal theory: giving and extracting meaning. In: Zaslavsky O, editor. Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education; 1999 Jul 25; Haifa (Israel): Israel Institute of Technology; 1999.
[8] Swinyard C. Reinventing the formal definition of limit: the case of Amy and Mike. J Math Behav. 2011;30(2):93–114.
[9] Cory B, Smith KW. Delving into limits of sequences. Math Teach. 2011;105(1):48–55.
[10] Roh KH. Students' images and their understanding to definitions of the limit of a sequence. Educ Stud Math. 2008;69(3):217–233.
[11] Roh KH. How to help students conceptualize the rigorous definition of the limit of a sequence. Probl Resour Issues Math Undergraduate Stud. 2010;20(6):473–487.
[12] Davis RB, Vinner S. The notion of limit: some seemingly unavoidable misconception stages. J Math Behav. 1986;5:284–303.
[13] Plass JL, Homer BD, Hayward EO. Design factors for educationally effective animations and simulations. J Comput Higher Educ. 2009;21(1):31–61.

[14] Kidron I, Zehavi N. The role of animation in teaching the limit concept. Int J Comput Algebra Math Educ. 2002;9(3):205–222.
[15] Roh KH, Lee YH. Development of students' understanding of the logic in the epsilon-N definition of limit. In: Ubuz B, editor. Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education; 2011 Jul 10; Ankara (Turkey): Middle East Technical University; 2011.
[16] Leung A. An epistemic model of task design in dynamic geometry environment. ZDM – Int J Math Educ. 2011;43:325–336.
[17] Bartolini Bussi MG, Mariotti MA. Semiotic mediation in the mathematics classroom: artifacts and signs after a Vygotskian perspective. In: English L, editor. Handbook of international research in mathematics education. 2nd ed. New York: Routledge; 2008. p. 750–787.
[18] Mariotti MA. ICT as opportunities for teaching-learning in a classroom: the semiotic potential of artefacts. In: Tso TY, editor. Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education; 2012 Jul 18; Taipei (Taiwan). Wesley Girls High School; 2012.
[19] Bartolini Bussi MG. Verbal interaction in mathematics classroom: a Vygotskian analysis. In: Steinbring H, Bartolini Bussi MG, Sierpinska A, editors. Language and communication in mathematics classroom. Reston, VA: NCTM; 1998. p. 65–84.
[20] Drijvers P, Boon P, Doorman M, Bokhove C, Tacoma S. Digital design: RME principles for designing online tasks. In: Margolinas C, editor. Task design in mathematics education: proceedings of ICMI Study 22. Oxford: University of Oxford; 2013. p. 53–60.
[21] Margolinas C, editor. Task design in mathematics education. Proceedings of ICMI Study 22; 2013 Jul 22; Oxford: University of Oxford.

Appendix 1 – Worksheets

Individual task-interview worksheets

(1) Consider $a_n = \frac{10\cos(n)}{n}$, use the applet to find the limit of and explain it step by step.
– For ε and N, which one depends on the other? Why?
(2) Use your own word to interpret$\lim_{n\to\infty} a_n = L$, where a_n is a sequence of real numbers.
– How would you relate the concept of infinity to the formal definition of the limit of a sequence?
(3) Is there any suggestion to the applet?
– Is there anything that is confusing?

Definition of limit of sequence:

$$\forall\varepsilon > 0, \exists N \in \mathbb{N} \text{ s. t. } |a_n - L| < \varepsilon \text{ for all } n \geq N$$

Consider $a_n = \frac{10\cos(n)}{n}$, use the applet to find the limit of a_n.
Lecture worksheets 1
Name: ________________________ Student ID: ________________________

(1) Use the applet to explain why $\lim_{n\to\infty} \frac{5}{n} = 0$ without mentioning infinity.
(2) Use the applet to explain why $\lim_{n\to\infty} n^2$ does not exist without mentioning infinity.
(3) Use the applet to explain why $\lim_{n\to\infty} \sin^2 n$ does not exist without mentioning infinity.
Lecture worksheets 2
Name: ________________________ Student ID: ________________________

For each case below, we form a rectangular shaded region on the applet with height 2ε and its left side at N. The box can be moved vertically for adjustment.

(4) Consider $a_n = \frac{5}{n}$,
(a) If ε is of the value below, can you find an N so that all the points with n greater than N are contained in the shaded region? If yes, what is it? If not, explain why.

- $\varepsilon = 2$
- $\varepsilon = 1$

(b) If N is of the value below, can you find an ε so that all the points with n greater than N are contained in the shaded region? If yes, what is it? If not, explain why.

- $N = 20$
- $N = 40$

Consider $a_n = n^2$

(a) If ε is of the value below, can you find an N so that all the points with n greater than N are contained in the shaded region? If yes, what is it? If not, explain why.

- $\varepsilon = 2$
- $\varepsilon = 1$

(b) If N is of the value below, can you find an ε so that all the points with n greater than N are contained in the shaded region? If yes, what is it? If not, explain why.

- $N = 20$
- $N = 40$

(5) Consider $a_n = \sin^2 n$

(a) If ε is of the value below, can you find an N so that all the points with n greater than N are contained in the shaded region? If yes, what is it? If not, explain why.

- $\varepsilon = 2$
- $\varepsilon = 1$

(b) If N is of the value below, can you find an ε so that all the points with n greater than N are contained in the shaded region? If yes, what is it? If not, explain why.

- $N = 20$
- $N = 40$

(6) Based on your findings in questions 1, 2 and 3, can you suggest a relationship between ε, N and a_n? Describe how they depend on each other and explain why.

Lecture worksheets 3.1

Name: ___________________________ Student ID: ___________________________

(1) In the applet, there is a rectangular shaded region (box) that can be dragged around. Besides the sequence a_n, there are four adjustable terms: ε, N, L and a.

(a) If we were to cover the tail part of the sequence with the box, values of some of the above terms may need to be adjusted accordingly. Please identify which one(s) may need to be adjusted and explain how they depend on each.

(b) Consider the inequality $|a_n - L| < \varepsilon$, where a_n represents a sequence. Is the inequality related to the shaded region? If so, please explain how the two are related.

Lecture worksheets 3.2

Name: ___________________________ Student ID: ___________________________

(2) Consider $a_n = \tan\left(2n \sin\left(\frac{1}{n}\right)\right)$.

(a) Guess the value of $\lim_{n\to\infty} a_n$. (You may use the applet to help with your guess.)

(b). Please justify your answer in part (a) by using ε and N in the applet.

Lecture worksheets 3.3

Name: ___________________________ Student ID: ___________________________

(3) Please suggest a definition of the existence of the limit (or convergence) of a sequence in terms of the terms: ε, N, L and a_n, Also, please avoid using words like 'tend to', 'approach' or 'infinity.